Integrable discretizations of the sine-Gordon equation

M. Boiti, F. Pempinelli, B. Prinari
Dipartimento di Fisica dell'Università and Sezione INFN,
73100 Lecce, Italy

A. Spire
Physique Mathématique et Théorique, CNRS-UMR5825,
Université Montpellier 2, 34095 Montpellier, France

March 10, 2002

Abstract

The inverse scattering theory for the sine-Gordon equation discretized in space and both in space and time is considered.

1 Introduction

In the framework of integrable nonlinear evolution equations, Hirota recovered almost 25 years ago the Bianchi superposition formula for the solutions of the sine-Gordon equation, showing that it can be considered as an integrable doubly discrete (both in space and time) version of the sine-Gordon equation [1]. He found also its Lax pair, Bäcklund transformations and N-soliton solutions. In the following, this study was generalized to include the discrete (in space or time) case [2], soliton solutions were obtained by using the dressing method [3], the related inverse spectral transform was studied [4] and the nature of the numerical instabilities of the solutions was discussed [5, 6].

Here we are interested in the spectral transform for the discrete and doubly discrete sine-Gordon equation with solutions decaying to 0 $(\operatorname{mod}\pi)$ at space infinity .

The aim of this paper is twofold.

From one side, we re-examine the scattering theory of the spectral problem introduced by Orfanidis in [2] which depends on a discretized space variable n. According to the usual scheme, we introduce two summation equations which define a couple of Jost solutions μ_n and ν_n characterized by their asymptotic behaviour, respectively, at $n \to -\infty$ and $n \to +\infty$. These summation equations are, as usual, used to define the spectral data. However, we show that, in contrast with the continuous case, in order to study the analytical properties of the Jost solution ν_n it is necessary to introduce an additional summation equation that only in the continuous limit reduces to the previous one. Then, we find the conditions on the potential that ensure the existence and analyticity properties of the Jost solutions, we formulate the inverse problem as a Riemann-Hilbert boundary value problem on the real axis and we derive the time evolution of the spectral data, paying special attention to the nature of the singularity of the time evolution of the solutions at $t = 0$.

From the other side, we show that both in the semi-discrete and doubly-discrete case, the equations of the matrix Orfanidis spectral problem can be decoupled and reduced to a scalar problem, giving as principal spectral problem the "exact discretization" of the Schrödinger operator introduced by Shabat [7]

$$\varphi_{n+2} = g_n \varphi_{n+1} - (1+\lambda)\varphi_n \tag{1.1}$$

and obtained iterating the Darboux transformations. Then, following a procedure analogous to that used in [8] for getting a 2+1 generalization of the sine-Gordon equations, we do not consider a Lax pair but a Lax triplet, i.e. we derive the discrete sine-Gordon equations as a compatibility condition of (1.1) with a pair of auxiliary spectral problems.

The spectral theory for the operator (1.1) was exhaustively studied in [9] for a potential g_n, not necessarily real, satisfying

$$\sum_{n=-\infty}^{+\infty} (1+|n|)|g_{n-1} - 2| < \infty. \tag{1.2}$$

and applied to a discrete version of the KdV. However, one must be advised that the solutions of the discrete sine-Gordon equations are related to the potential g_n via an equation that can be considered a discretized version of a Riccati equation and can be solved in terms of a continous fraction. Moreover, the potential g_n is complex and the problem of characterizing the spectral data is left open.

By using this theory we introduce the spectral data, we find their time evolution and, therefore, according to the usual inverse scattering scheme, the Cauchy initial value problem is linearized. In particular, we show that the time evolution is discontinuous at the initial time $t = 0$.

We are also able to find the couple of doubly discrete auxiliary spectral operators that in triplet with the doubly discrete Schrödinger operator gives the Hirota-Bianchi fourth order doubly discrete sine-Gordon equation.

The whole theory can be trivially extended to the sinh-Gordon case. Then, the potential g_n is real and the characterization equations for the spectral data are easily obtained in analogy with the continuous case.

2 Orfanidis Lax pair

Let us consider the Lax pair proposed by Orfanidis [2]

$$\chi_{n+1,m} = M_{n,m}\chi_{n,m} \tag{2.1}$$

$$\chi_{n,m+1} = N_{n,m}\chi_{n,m} \tag{2.2}$$

with

$$M_{n,m} = \begin{pmatrix} e^{-i(\theta_{n+1,m}-\theta_{n,m})/2} & ik \\ ik & e^{i(\theta_{n+1,m}-\theta_{n,m})/2} \end{pmatrix} \tag{2.3}$$

$$N_{nm} = \begin{pmatrix} 1 & \frac{\gamma}{ik}\, e^{-i(\theta_{n,m+1}+\theta_{n,m})/2} \\ \frac{\gamma}{ik}\, e^{i(\theta_{n,m+1}+\theta_{n,m})/2} & 1 \end{pmatrix} \tag{2.4}$$

with $\theta_{n,m}$ real, k the spectral parameter and γ a real constant. The compatibility condition

$$F\chi_{n+1,m} = E\chi_{n,m+1}$$

where E shifts n and F shifts m gives by inserting in it (2.1) and (2.2) $M_{n,m+1}N_{n,m} = N_{n+1,m}M_{n,m}$. From it we have the Bianchi-Hirota equation

$$\sin\left(\frac{\theta_{n+1,m+1} - \theta_{n+1,m} - \theta_{n,m+1} + \theta_{n,m}}{4}\right) = \gamma \sin\left(\frac{\theta_{n+1,m+1} + \theta_{n+1,m} + \theta_{n,m+1} + \theta_{n,m}}{4}\right). \tag{2.5}$$

If we introduce

$$\theta_{i,j} = \theta_i(t + \tau j), \qquad t = m\tau, \tag{2.6}$$

change the constant γ as follows

$$\gamma \to \tau\gamma \tag{2.7}$$

and take the limit $\tau \to 0$ and $m \to \infty$ at t fixed we get the semi-discrete sine-Gordon equation in the light cone coordinates

$$\partial_t\theta_{n+1} - \partial_t\theta_n = 4\gamma \sin\frac{1}{2}\left(\theta_{n+1} + \theta_n\right). \tag{2.8}$$

Notice that this equation can be obtained as compatibility condition of the Lax pair

$$\chi_{n+1} = M_n\chi_n \tag{2.9}$$

$$\partial_t\chi_n = N_n\chi_n \tag{2.10}$$

with

$$M_n = \begin{pmatrix} e^{-i(\theta_{n+1}-\theta_n)/2} & ik \\ ik & e^{i(\theta_{n+1}-\theta_n)/2} \end{pmatrix} \tag{2.11}$$

$$N_n = \begin{pmatrix} 0 & \frac{\gamma}{ik}\, e^{-i\theta_n} \\ \frac{\gamma}{ik}\, e^{i\theta_n} & 0 \end{pmatrix} \tag{2.12}$$

Notice also that in both cases, doubly discrete and space discrete cases, the principal spectral problem (2.1) is the same.

3 Direct scattering problem

3.1 Jost solutions and summation equations

Let us rewrite the spectral problem (2.1) as

$$\chi_{n+1} - (\mathbf{1} + ik\sigma_1)\chi_n = Q_n\chi_n \tag{3.1}$$

where the Pauli σ matrices are defines as usual, i.e.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2}$$

and

$$Q_n = e^{-\frac{i}{2}\sigma_3(\theta_{n+1}-\theta_n)} - \mathbf{1} \tag{3.3}$$

and let us consider it for $k = k_{\text{Re}}$.

If G_n is a Green function, i.e. if G_n satisfies

$$G_{n+1} - (1 + ik\sigma_1)G_n = \delta_{n,0}\mathbf{1} \tag{3.4}$$

a matrix solution of (3.1) is given by the solution of the following summation equation

$$\chi_n = w_n + \sum_{j=-\infty}^{+\infty} G_{n-j} Q_j \chi_j \tag{3.5}$$

where w_n is a solution of the homogeneous equation

$$w_{n+1} - (\mathbf{1} + ik\sigma_1)w_n = 0. \tag{3.6}$$

We represent G_n and $\delta_{n,0}$ as Fourier integrals

$$G_n = \frac{1}{2\pi i}\oint_{|p|=R} p^{n-1}\widehat{G}(p)dp, \tag{3.7}$$

$$\delta_{n,0} = \frac{1}{2\pi i}\oint_{|p|=R} p^{n-1}dp. \tag{3.8}$$

Inserting into (3.4) we get

$$((p-1)\mathbf{1} - ik\sigma_1)\widehat{G}(p) = \mathbf{1} \tag{3.9}$$

and therefore

$$\widehat{G}(p) = \frac{1}{(p-1)^2 + k^2}((p-1)\mathbf{1} + ik\sigma_1). \tag{3.10}$$

Notice that $\widehat{G}(p)$ has a pole at $p = 1 \pm ik$ and notice also that at large p

$$\left(\mathbf{1} - \frac{\mathbf{1} + ik\sigma_1}{p}\right)^{-1} = \sum_{j=0}^{+\infty} \frac{(\mathbf{1} + ik\sigma_1)^j}{p^j} \tag{3.11}$$

and therefore at large p

$$\widehat{G}(p) = \sum_{j=0}^{+\infty} \frac{(\mathbf{1} + ik\sigma_1)^j}{p^{j+1}}. \tag{3.12}$$

We consider $k = k_{\text{Re}}$ and for different choices of R, i.e. for $R > 1$ and $R < 1$, we get different Green functions, G and H, and, correspondingly, different summation equations defining different Jost solutions, ϕ and ψ.

For $R > 1 + k^2$ the integrand in (3.7) inside the disk $|p| \le R$ contains a pole at $p = 0$ for $n \le 0$ and the poles $p = 1 \pm ik$ and it is analytic outside the disk. The integral can be computed evaluating the residuum at $p = \infty$ by using (3.12) and we get

$$G_n = \Theta(n-1)\,(\mathbf{1} + ik\sigma_1)^{n-1} \tag{3.13}$$

where $\Theta(n)$ is the discrete version of the Heaviside function

$$\Theta(n) = \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases}. \tag{3.14}$$

Therefore

$$\phi_n = w_n + \sum_{j=-\infty}^{n-1} (\mathbf{1} + ik\sigma_1)^{n-j-1} Q_j \phi_j. \tag{3.15}$$

If $R < 1$ we have to subtract to the previous integral the residua at the poles $p = 1 \pm ik$. We have

$$H_n = \Theta(n-1)\,(\mathbf{1} + ik\sigma_1)^{n-1} - \frac{1}{2}(\mathbf{1} + \sigma_1)(1 + ik)^{n-1} - \frac{1}{2}(\mathbf{1} - \sigma_1)(1 - ik)^{n-1}. \tag{3.16}$$

Since

$$(\mathbf{1} + ik\sigma_1)^n = \frac{1}{2}(\mathbf{1} + \sigma_1)(1 + ik)^n + \frac{1}{2}(\mathbf{1} - \sigma_1)(1 - ik)^n \tag{3.17}$$

we have

$$H_n = -\Theta(-n)\,(\mathbf{1} + ik\sigma_1)^{n-1} \tag{3.18}$$

and the summation equation becomes

$$\psi_n = w_n - \sum_{j=n}^{+\infty} (\mathbf{1} + ik\sigma_1)^{n-j-1} Q_j \psi_j. \tag{3.19}$$

It is convenient to choose

$$w_n = (\mathbf{1} - i\sigma_2)(\mathbf{1} + ik\sigma_3)^n \tag{3.20}$$

so that ϕ_n and ψ_n satisfy the following boundary conditions

$$\phi_n \sim (\mathbf{1} - i\sigma_2)\,(\mathbf{1} + ik\sigma_3)^n \qquad n \to -\infty \tag{3.21}$$

$$\psi_n \sim (\mathbf{1} - i\sigma_2)\,(\mathbf{1} + ik\sigma_3)^n \qquad n \to +\infty. \tag{3.22}$$

In order to study the analytical properties of the Jost solutions with respect to the spectral parameter k, it is necessary to introduce the modified matrix Jost solutions

$$\mu_n = \phi_n\,(\mathbf{1} + ik\sigma_3)^{-n} \tag{3.23}$$

$$\nu_n = \psi_n\,(\mathbf{1} + ik\sigma_3)^{-n} \tag{3.24}$$

and, then, to consider separately the two columns of these matrices, which we denote as follows

$$\mu_n = (\mu_n^-, \mu_n^+), \quad \nu_n = (\nu_n^+, \nu_n^-). \tag{3.25}$$

They satisfy the difference equations

$$(1 \mp ik)\,\mu_{n+1}^{\pm} = [\mathbf{1} + ik\sigma_1 + Q_n]\,\mu_n^{\pm} \tag{3.26}$$

$$(1 \pm ik)\,\nu_{n+1}^{\pm} = [\mathbf{1} + ik\sigma_1 + Q_n]\,\nu_n^{\pm} \tag{3.27}$$

and have constant asymptotic behaviour

$$\mu_n \sim (\mathbf{1} - i\sigma_2) \qquad n \to -\infty \tag{3.28}$$

$$\nu_n \sim (\mathbf{1} - i\sigma_2) \qquad n \to +\infty. \tag{3.29}$$

3.2 Existence and analyticity of the Jost solutions $\mu^{\pm}$

We will show that the Jost solutions $\mu_n^{\pm}(k)$ defined by the summation equations

$$\mu_n^+(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{j=-\infty}^{n-1} \frac{(\mathbf{1} + ik\sigma_1)^{n-j-1}}{(1-ik)^{n-j}} Q_j \mu_j^+(k) \tag{3.30}$$

$$\mu_n^-(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{j=-\infty}^{n-1} \frac{(\mathbf{1} + ik\sigma_1)^{n-j-1}}{(1+ik)^{n-j}} Q_j \mu_j^-(k) \tag{3.31}$$

are analytic, correspondingly, in the upper half plane and in the lower half plane of the spectral parameter k and continuous for $k_{\mathrm{Im}} \geq 0$ and $k_{\mathrm{Im}} \leq 0$.

Let us write the solution of the integral equation (3.30) in the form of a Neumann series

$$\mu_n^+(k) = \sum_{\ell=0}^{+\infty} C_n^{\ell}(k) \qquad C_n^{\ell}(k) = \begin{pmatrix} C_n^{\ell,(1)}(k) \\ C_n^{\ell,(2)}(k) \end{pmatrix} \tag{3.32}$$

where

$$C_n^0(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad C_n^{\ell+1}(k) = \sum_{j=-\infty}^{n-1} \frac{(\mathbf{1} + ik\sigma_1)^{n-j-1}}{(1-ik)^{n-j}} Q_j C_j^{\ell}(k) \tag{3.33}$$

or, in component form, taking into account the identity (3.17)

$$\begin{aligned} C_n^{\ell+1,(1)}(k) = \frac{1}{2(1-ik)} \sum_{j=-\infty}^{n-1} \Bigg\{ & \left[1 + \left(\frac{1+ik}{1-ik}\right)^{n-j-1}\right] \left(e^{-\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) C_j^{\ell,(1)}(k) \\ & - \left[1 - \left(\frac{1+ik}{1-ik}\right)^{n-j-1}\right] \left(e^{\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) C_j^{\ell,(2)}(k) \Bigg\} \end{aligned} \tag{3.34}$$

$$\begin{aligned} C_n^{\ell+1,(2)}(k) = \frac{1}{2(1-ik)} \sum_{j=-\infty}^{n-1} \Bigg\{ & - \left[1 - \left(\frac{1+ik}{1-ik}\right)^{n-j-1}\right] \left(e^{-\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) C_j^{\ell,(1)}(k) \\ & + \left[1 + \left(\frac{1+ik}{1-ik}\right)^{n-j-1}\right] \left(e^{\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) C_j^{\ell,(2)}(k) \Bigg\}. \end{aligned} \tag{3.35}$$

The series (3.32) is formally a solution of the discrete integral equation (3.30). One can prove by induction on $\ell \in \mathbb{N}$ that for $k_{\mathrm{Im}} \geq 0$

$$\left| C_n^{\ell,(j)}(k) \right| \leq \frac{2^{\ell}}{|1-ik|^{\ell}} \frac{1}{\ell!} \left[\sum_{l=-\infty}^{n-1} \left| e^{\frac{i}{2}(\theta_{l+1}-\theta_l)} - 1 \right| \right]^{\ell}, \qquad j = 1, 2. \tag{3.36}$$

Indeed, for each component $j = 1, 2$ we have

$$\begin{aligned} \left| C_n^{\ell+1,(j)}(k) \right| \leq \frac{1}{2\,|1-ik|} \sum_{l=-\infty}^{n-1} \left| e^{-\frac{i}{2}(\theta_{l+1}-\theta_l)} - 1 \right| \Bigg[& \left| 1 + \left(\frac{1+ik}{1-ik}\right)^{n-l-1} \right| \left| C_l^{\ell,(1)}(k) \right| \\ & + \left| 1 - \left(\frac{1+ik}{1-ik}\right)^{n-l-1} \right| \left| C_l^{\ell,(2)}(k) \right| \Bigg] \end{aligned}$$

where we used that the potential θ_n is real. Moreover

$$\left|1 \pm \left(\frac{1+ik}{1-ik}\right)^{n-l-1}\right| \leq 1 + \left|\frac{1+ik}{1-ik}\right|^{n-l-1} \leq 2 \tag{3.37}$$

since $k_{\mathrm{Im}} > 0$ and $n-l-1 \geq 0$. Therefore we have

$$\left|C_n^{\ell+1,(j)}(k)\right| \leq \frac{1}{|1-ik|}\sum_{l=-\infty}^{n-1}\left|e^{-\frac{i}{2}(\theta_{l+1}-\theta_l)}-1\right|\left[\left|C_l^{\ell,(1)}(k)\right| + \left|C_l^{\ell,(2)}(k)\right|\right] \tag{3.38}$$

and we can use the inductive hypothesis to get

$$\left|C_n^{\ell+1,(j)}(k)\right| \leq \frac{1}{\ell!}\frac{2^{\ell+1}}{|1-ik|^{\ell+1}}\sum_{l=-\infty}^{n-1}\left|e^{-\frac{i}{2}(\theta_{l+1}-\theta_l)}-1\right|\left(\sum_{l_1=-\infty}^{\ell-1}\left|e^{-\frac{i}{2}\left(\theta_{l_1+1}-\theta_{l_1}\right)}-1\right|\right)^{\ell}. \tag{3.39}$$

The use of the summation by parts formula, according to which for any real non-negative sequence $\{b_j\}_{j=-\infty}^{+\infty}$ such that the series $\sum_{j=-\infty}^{+\infty} b_j$ is convergent and for any $m \in \mathbb{N}_0$

$$\sum_{k=-\infty}^{n} b_k \left(\sum_{j=-\infty}^{k-1} b_j\right)^m \leq \frac{1}{m+1}\left(\sum_{j=-\infty}^{n-1} b_j\right)^{m+1}, \tag{3.40}$$

completes the proof of (3.36).

Therefore for $k_{\mathrm{Im}} \geq 0$

$$\left|C_n^{\ell,(j)}(k)\right| \leq \frac{P(n)^{\ell}}{\ell!}, \qquad j=1,2 \tag{3.41}$$

where

$$P(n) = 2\sum_{j=-\infty}^{n-1}\left|e^{\frac{i}{2}(\theta_{j+1}-\theta_j)}-1\right| \tag{3.42}$$

and we conclude that the Neumann series (3.32) for $\mu_n^+(k)$ is uniformly convergent (with respect to n and k in the upper half-plane) and

$$\left|\mu_n^{+,(j)}(k)\right| \leq e^{P(\infty)} \qquad j=1,2 \tag{3.43}$$

provided the potential θ_n is such that

$$P(\infty) = 2\sum_{j=-\infty}^{+\infty}\left|e^{\frac{i}{2}(\theta_{j+1}-\theta_j)}-1\right| < +\infty. \tag{3.44}$$

In the same way one can prove that μ_n^- is analytic for $k_{\mathrm{Im}} < 0$ and continuous for $k_{\mathrm{Im}} \leq 0$.

3.3 Existence and analyticity of the Jost solutions $\nu^{\pm}$

Let us now consider the summation equations defining the Jost solutions $\nu^{\pm}$, which can be rewritten as

$$\nu_n^+(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{1+ik} \sum_{j=n}^{+\infty} \frac{(\mathbf{1} - ik\sigma_1)^{j-n+1}}{(1-ik)^{j-n+1}} Q_j \nu_j^+(k) \tag{3.45}$$

$$\nu_n^-(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{1}{1-ik} \sum_{j=n}^{+\infty} \frac{(\mathbf{1} - ik\sigma_1)^{j-n+1}}{(1+ik)^{j-n+1}} Q_j \nu_j^-(k) \tag{3.46}$$

showing explicitly that their kernel is singular at $k = \pm i$, i.e. in both the lower and upper half planes of k. This dissymmetry of the summation equations for the Jost solutions $\nu^\pm$ with respect to $\mu^\pm$ is peculiar of the discrete case, since it disappears in the continuous limit. In fact, in order to study the analytic properties of the Jost solutions one needs to define the same solutions by introducing alternative summation equations, which can be obtained by exploiting the special structure of the potential Q_n and the symmetry property of the spectral problem (3.1) for $n \to -n$.

By using the relation

$$Q_n + \sigma_1 Q_n \sigma_1 + \sigma_1 Q_n \sigma_1 Q_n = 0 \tag{3.47}$$

one can easily verify that

$$(\mathbf{1} + ik\sigma_1 + Q_n)^{-1} = \frac{1}{1+k^2}(\mathbf{1} - ik\sigma_1 + \sigma_1 Q_n \sigma_1) \tag{3.48}$$

and therefore the spectral problem (3.1) can be rewritten as

$$\frac{1}{1+k^2}(\mathbf{1} - ik\sigma_1 + \sigma_1 Q_n \sigma_1)\chi_{n+1} = \chi_n. \tag{3.49}$$

If we introduce

$$\chi_{n+1} = (1+k^2)^n \xi_{-n} \tag{3.50}$$

we have

$$\xi_{n+1} = (\mathbf{1} - ik\sigma_1 + \sigma_1 Q_{-n} \sigma_1)\xi_n \tag{3.51}$$

and, consequently, using the same procedure we followed above for ϕ_n we get

$$\xi_n = \xi_{0n} + \sum_{j=-\infty}^{n-1} (\mathbf{1} - ik\sigma_1)^{n-j-1} \sigma_1 Q_{-j} \sigma_1 \xi_j \tag{3.52}$$

and coming back to χ_n

$$\chi_n = w_n + \sum_{\ell=n+1}^{+\infty} (1+k^2)^{n-\ell} (\mathbf{1} - ik\sigma_1)^{\ell-n-1} \sigma_1 Q_{\ell-1} \sigma_1 \chi_\ell \tag{3.53}$$

where we choose w_n as in (3.20).

One can check explicitly that ψ_n defined in (3.19) satisfies this summation equation by moving in (3.19) the $j = n$ term of the sum from the right side to the left side, by applying from the left $(\mathbf{1} + ik\sigma_1 + Q_n)^{-1}(\mathbf{1} + ik\sigma_1)$ and then by using on the right (3.48), (3.47) and the spectral equation for ψ_n.

Therefore from (3.53) by considering the transformation (3.24) we get the following alternative summation equations for $\nu^{\pm}(k)$

$$\nu_n^+(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{j=n+1}^{+\infty} \frac{(\mathbf{1} - ik\sigma_1)^{j-n-1}}{(1-ik)^{j-n}} \sigma_1 Q_{j-1} \sigma_1 \nu_j^+(k) \tag{3.54}$$

$$\nu_n^-(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{j=n+1}^{+\infty} \frac{(\mathbf{1} - ik\sigma_1)^{j-n-1}}{(1+ik)^{j-n}} \sigma_1 Q_{j-1} \sigma_1 \nu_j^-(k). \tag{3.55}$$

Following a procedure analogous to that one used for the $\mu^{\pm}(k)$ one can prove that the $\nu^{\pm}(k)$ are also analytic, correspondingly, in the upper half plane and in the lower half plane of the spectral parameter k and continuous for $k_{\mathrm{Im}} \geq 0$ and $k_{\mathrm{Im}} \leq 0$ provided the potential satisfies the condition (3.44).

Finally, let us note that, taking into account the explicit expression of the Green's functions, one can easily obtain the asymptotic behavior of the Jost solutions at large k

$$\mu_n^+(k) \sim \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \nu_n^+(k) \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |k| \to \infty \qquad k_{\mathrm{Im}} > 0 \tag{3.56}$$

$$\mu_n^-(k) \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \nu_n^-(k) \sim \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad |k| \to \infty \qquad k_{\mathrm{Im}} < 0. \tag{3.57}$$

3.4 Scattering data

Let us define the Wronskian of any two vectors v and w as

$$W(v, w) = \det(v, w). \tag{3.58}$$

The vector-valued sequences v_n and w_n are linearly independent if $W(v_n, w_n) \neq 0$ for all n.

In particular, if v_n and w_n are any two solutions of the scattering problem (3.1), their Wronskian satisfies the recursive relation

$$W(v_{n+1}, w_{n+1}) = \left(1 + k^2\right) W(v_n, w_n). \tag{3.59}$$

Hence, for any positive integer j

$$\begin{aligned} W\left(\phi_n^+(k), \phi_n^-(k)\right) &= \left(1 + k^2\right)^j W\left(\phi_{n-j}^+(k), \phi_{n-j}^-(k)\right) \\ &= \left(1 + k^2\right)^n W\left(\mu_{n-j}^+(k), \mu_{n-j}^-(k)\right) \end{aligned} \tag{3.60}$$

and

$$\begin{aligned} W\left(\psi_n^+(k), \psi_n^-(k)\right) &= \left(1 + k^2\right)^{-j} W\left(\psi_{n+j}^+(k), \psi_{n+j}^-(k)\right) \\ &= \left(1 + k^2\right)^n W\left(\nu_{n+j}^+(k), \nu_{n+j}^-(k)\right). \end{aligned} \tag{3.61}$$

Taking into account (3.28) and (3.29), for $k = k_{\mathrm{Re}}$ in the limit $j \to \infty$ we get

$$W\left(\phi_n^+(k), \phi_n^-(k)\right) = -2\left(1 + k^2\right)^n, \qquad W\left(\psi_n^+(k), \psi_n^-(k)\right) = 2\left(1 + k^2\right)^n \tag{3.62}$$

which shows that the eigenfunctions ϕ_n^+ and ϕ_n^- are linearly independent, as well as ψ_n^+ and ψ_n^-. Since the discrete scattering problem (3.1) is a second-order difference equation, there are at most two linearly independent solutions for any fixed value of k and consequently we can write ϕ_n^+ and ϕ_n^- as linear combination of ψ_n^+ and ψ_n^- or vice-versa. The coefficients of this linear combinations depend on k. Hence the relations ($k = k_{\rm Re}$)

$$\phi_n^{\pm}(k) = b^{\pm}(k)\psi_n^{\pm}(k) + a^{\pm}(k)\psi_n^{\mp}(k) \tag{3.63}$$

which define the scattering data $a^{\pm}(k)$, $b^{\pm}(k)$.

In terms of the Jost solutions, they can also be written as

$$\frac{\mu_n^{\pm}(k)}{a^{\pm}(k)} = \rho^{\pm}(k)\left(\frac{1 \pm ik}{1 \mp ik}\right)^n \nu_n^{\pm}(k) + \nu_n^{\mp}(k) \tag{3.64}$$

where we introduced the reflection coefficients

$$\rho^{\pm}(k) = \frac{b^{\pm}(k)}{a^{\pm}(k)}. \tag{3.65}$$

Notice that from (3.64), by using (3.29), we can get the asymptotics of $\mu_n^{\pm}$ for $n \to +\infty$

$$\mu_n^{\pm}(k) \sim b^{\pm}(k)\left(\frac{1 \pm ik}{1 \mp ik}\right)^n \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix} + a^{\pm}(k)\begin{pmatrix} \mp 1 \\ 1 \end{pmatrix}. \tag{3.66}$$

Calculating $W\left(\phi_n^+(k), \phi_n^-(k)\right)$ using (3.63) results in

$$W\left(\phi_n^+(k), \phi_n^-(k)\right) = -\left(a^+(k)a^-(k) - b^+(k)b^-(k)\right) W\left(\psi_n^+(k), \psi_n^-(k)\right) \tag{3.67}$$

and taking into account (3.62) yields

$$a^+(k)a^-(k) - b^+(k)b^-(k) = 1 \qquad k \in \mathbb{R}. \tag{3.68}$$

Moreover,

$$W\left(\phi_n^{\pm}(k), \psi_n^{\pm}(k)\right) = \mp 2\left(1 + k^2\right)^n a^{\pm}(k) \tag{3.69}$$

or, equivalently,

$$a^{\pm}(k) = \mp\frac{1}{2} W\left(\mu_n^{\pm}(k), \nu_n^{\pm}(k)\right) \tag{3.70}$$

which proves, from one side, that $a^+(k)$ can be analytically continued in the upper half plane (respectively $a^-(k)$ can be analytically continued in the lower half plane) and, from the other side, that the zeros of $a^+(k)$ in the upper half plane (respectively of $a^-(k)$) correspond to bound states of the scattering problem.

We note that for real potentials θ_n, if $\chi_n(k)$ satisfies the scattering problem (3.1), then

$$\tilde{\chi}_n(k) = i\sigma_2 \chi_n^*(k^*) \tag{3.71}$$

satisfies the same equation. Taking into account the boundary conditions (3.21) and (3.22), we conclude that if the potential θ_n is real the Jost solutions obey the symmetry conditions

$$\phi_n^-(k) = i\sigma_2\left(\phi_n^+(k^*)\right)^* \qquad \psi_n^-(k) = -i\sigma_2\left(\psi_n^+(k^*)\right)^*. \tag{3.72}$$

Moreover, one can easily show by using (3.69) that

$$a^-(k) = \left(a^+(k^*)\right)^* \tag{3.73}$$

and by using (3.63) for $k = k_{\text{Re}}$ that

$$b^-(k) = -\left(b^+(k)\right)^* . \tag{3.74}$$

The scattering coefficients can also be given as explicit sum of the Jost solutions. The formulae are derived as follows. First, we rewrite for $k = k_{\text{Re}}$ the summation equation (3.30) as

$$\begin{aligned}\mu_n^+(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} &+ \sum_{j=-\infty}^{+\infty} \frac{(1+ik\sigma_1)^{n-j-1}}{(1-ik)^{n-j}} Q_j \mu_j^+(k) \\ &- \frac{1}{(1-ik)} \sum_{j=n}^{+\infty} \frac{(1-ik\sigma_1)^{j-n+1}}{(1+ik)^{j-n+1}} Q_j \mu_j^+(k)\end{aligned} \tag{3.75}$$

and, then, using (3.46) we obtain

$$\begin{aligned}\mu_n^+(k) - \nu_n^-(k)a^+(k) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \left(1 - a^+(k)\right) &+ \sum_{j=-\infty}^{+\infty} \frac{(1+ik\sigma_1)^{n-j-1}}{(1-ik)^{n-j}} Q_j \mu_j^+(k) \\ &- \frac{1}{(1-ik)} \sum_{j=n}^{+\infty} \frac{(1-ik\sigma_1)^{j-n+1}}{(1+ik)^{j-n+1}} Q_j \left[\mu_j^+(k) - \nu_j^-(k)a^+(k)\right] .\end{aligned} \tag{3.76}$$

Inserting in it (3.64) we have

$$\begin{aligned}&b^+(k) \left(\frac{1+ik}{1-ik}\right)^n \left[\nu_n^+(k) - \frac{1}{(1+ik)} \sum_{j=n}^{+\infty} \frac{(1-ik\sigma_1)^{j-n+1}}{(1-ik)^{j-n+1}} Q_j \nu_j^+(k)\right] \\ &\quad = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \left(1 - a^+(k)\right) + \sum_{j=-\infty}^{+\infty} \frac{(1+ik\sigma_1)^{n-j-1}}{(1-ik)^{n-j}} Q_j \mu_j^+(k).\end{aligned} \tag{3.77}$$

From (3.45) it follows that the term in square brackets in the left-hand side is $(1\ ,1)^T$ and therefore we obtain using (3.17)

$$\begin{aligned}a^+(k) = 1 + \frac{1}{2(1-ik)} \sum_{j=-\infty}^{+\infty} &\left[-\left(e^{-\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) \mu_j^{+,(1)}(k)\right. \\ &\left. + \left(e^{\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) \mu_j^{+,(2)}(k)\right]\end{aligned} \tag{3.78}$$

$$\begin{aligned}b^+(k) = \frac{1}{2(1-ik)} \sum_{j=-\infty}^{+\infty} &\left(\frac{1+ik}{1-ik}\right)^{-j-1} \left[\left(e^{-\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) \mu_j^{+,(1)}(k)\right. \\ &\left. + \left(e^{\frac{i}{2}(\theta_{j+1}-\theta_j)} - 1\right) \mu_j^{+,(2)}(k)\right] .\end{aligned} \tag{3.79}$$

These (discrete version of) integral representations, together with the symmetry relations (3.73) and (3.74) completely determine the scattering coefficients.

The discrete scattering problem (3.1) can possess discrete eigenvalues (bound states). These occur whenever $a^+(k_j^+) = 0$ for some k_j^+ in the upper half plane or $a^-(k_\ell^-) = 0$ for some k_ℓ^- in the lower half plane. Indeed, for such values of the spectral parameter from

(3.69) it follows that $W\left(\phi_n^+(k_j^+),\psi_n^+(k_j^+)\right)=0$ and $W\left(\phi_n^-(k_\ell^-),\psi_n^-(k_\ell^-)\right)=0$ and therefore the eigenfunctions are linearly dependent, i.e.

$$\phi_n^+(k_j^+)=b_j^+\psi_n^+(k_j^+) \qquad j=1,\ldots,J^+ \tag{3.80}$$

$$\phi_n^-(k_\ell^-)=b_\ell^-\psi_n^-(k_\ell^-) \qquad \ell=1,\ldots,J^- \tag{3.81}$$

for some complex constants $b_j^\pm$. In terms of the modified Jost solutions (3.80), (3.81) can be written as

$$\mu_n^+(k_j^+)=b_j^+\left(\frac{1+ik_j^+}{1-ik_j^+}\right)^n\nu_n^+(k_j^+), \qquad \mu_n^-(k_\ell^-)=b_\ell^-\left(\frac{1-ik_\ell^-}{1+ik_\ell^-}\right)^n\nu_n^-(k_\ell^-). \tag{3.82}$$

Note that when the potential θ_n is real, (3.73) implies that if k_j^+ is a zero of $a^+(k)$ in the upper half-plane, then $k_j^-=\left(k_j^+\right)^*$ is a zero of $a^-(k)$ in the lower half-plane and therefore $J^+=J^-$.

4 Inverse Scattering problem

Let us assume $a^+(k)$ has J^+ simple zeros $\left\{k_j^+\right\}_{j=1}^{J^+}$ in the upper half plane and $a^-(k)$ has J^- simple zeros $\left\{k_j^-\right\}_{j=1}^{J^-}$ in the lower half plane. Then, by (3.82) it follows

$$\operatorname{Res}\left(\frac{\mu_n^+(k)}{a^+(k)};k_j^+\right)=\mu_n^+(k_j^+)\left(\left.\frac{da^+}{dk}\right|_{k=k_j^+}\right)^{-1}=C_j^+\left(\frac{1+ik_j^+}{1-ik_j^+}\right)^n\nu_n^+(k_j^+) \tag{4.1}$$

$$\operatorname{Res}\left(\frac{\mu_n^-(k)}{a^-(k)};k_j^-\right)=\mu_n^-(k_j^-)\left(\left.\frac{da^-}{dk}\right|_{k=k_j^-}\right)^{-1}=C_j^-\left(\frac{1-ik_j^-}{1+ik_j^-}\right)^n\nu_n^-(k_j^-) \tag{4.2}$$

where

$$C_j^+=b_j^+\left(\left.\frac{da^+}{dk}\right|_{k=k_j^+}\right)^{-1}, \qquad C_j^-=b_j^-\left(\left.\frac{da^-}{dk}\right|_{k=k_j^-}\right)^{-1}. \tag{4.3}$$

Taking into account the analytic properties of the Jost solutions and of the scattering coefficients $a^\pm(k)$, as well as the asymptotic behavior (3.56), we can use the Cauchy-Green formula to obtain from the “jump” conditions (3.64)

$$\begin{aligned}\nu_n^-(k)=&\begin{pmatrix}-1\\1\end{pmatrix}+\sum_{j=1}^{J^+}\left(\frac{1+ik_j^+}{1-ik_j^+}\right)^n\frac{C_j^+}{k-k_j^+}\nu_n^+(k_j^+)\\&+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\rho^+(\zeta)}{\zeta-k+i0}\left(\frac{1+i\zeta}{1-i\zeta}\right)^n\nu_n^+(\zeta)d\zeta\end{aligned} \tag{4.4}$$

$$\begin{aligned}\nu_n^+(k)=&\begin{pmatrix}1\\1\end{pmatrix}+\sum_{j=1}^{J^-}\left(\frac{1-ik_j^-}{1+ik_j^-}\right)^n\frac{C_j^-}{k-k_j^-}\nu_n^-(k_j^-)\\&-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\rho^-(\zeta)}{\zeta-k-i0}\left(\frac{1-i\zeta}{1+i\zeta}\right)^n\nu_n^-(\zeta)d\zeta.\end{aligned} \tag{4.5}$$

Equations (4.4) and (4.5) define a Riemann-Hilbert problem on the line which, in principle, allows one to solve the inverse problem, i.e. to reconstruct the Jost solutions from the scattering data.

In order to complete the inverse problem, we need to reconstruct the potential from the scattering data. Let us rewrite the difference equations (3.27) in matrix form

$$[1 + ik\sigma_3]\,\nu_{n+1} = [1 + ik\sigma_1 + Q_n]\,\nu_n \tag{4.6}$$

and let us write the asymptotic expansion at large k of $\nu_n = (\nu_n^+, \nu_n^-)$

$$\nu_n(k) = \nu_n^{(0)} + \frac{\nu_n^{(-1)}}{ik} + \dots \tag{4.7}$$

where, strictly speaking, the expansion of the first column is performed for $k_{\rm Im} > 0$ while the expansion of the second one is performed for $k_{\rm Im} < 0$. Inserting this expansion into (4.6) and taking into account that, according to (3.56) and (3.57), $\nu_n^{(0)} = 1 - i\sigma_2$ yields

$$Q_n = \frac{1}{2}\left[\nu_{n+1}^{(-1)}\sigma_3 - \sigma_1\nu_n^{(-1)}\right](1 + i\sigma_2)\,. \tag{4.8}$$

5 Time evolution of spectral data

The time evolution of the Jost solutions and, consequently, of the scattering data is fixed by (2.10).

Since the matrix $\mu_n(1 + i\sigma_3 k)^n$ is a general solution of (2.9) there exists a $\Omega(t;k)$ such that

$$\chi_n(t;k) = \mu_n(t;k)(1 + i\sigma_3 k)^n\Omega(t;k), \tag{5.1}$$

satisfies both (2.9) and (2.10).

Inserting (5.1) into (2.10), if $\theta_n \to r\pi$ ($r \in \mathbb{N}$) for $n \to \pm\infty$, by evaluating the limit $n \to -\infty$ we get, recalling (3.28),

$$\Omega(t;k) = \exp\left(-i\sigma_3\frac{\eta\gamma}{k}t\right) \tag{5.2}$$

where $\eta = (-1)^r$.

Then, we evaluate the limit for $n \to +\infty$ by using (3.66) and we obtain the time evolution of spectral data ($k = k_{\rm Re}$)

$$b_t^\pm(k) = \mp\frac{2i\eta\gamma}{k \mp i0\eta\gamma t}b^\pm(k), \qquad a_t^\pm(k) = 0 \tag{5.3}$$

i.e.

$$b^\pm(k,t) = b^\pm(k,0)e^{\mp 2i\frac{\eta\gamma}{k}t}, \qquad a^\pm(k,t) = a^\pm(k,0). \tag{5.4}$$

For the discrete spectral data one gets that the locations of poles at $k = k_j^\pm$ are fixed while

$$C_j^\pm(t) = C_j^\pm(0)e^{\mp 2i\frac{\eta\gamma}{k_j^\pm}t}. \tag{5.5}$$

Notice that the switching on of the time evolution introduces a singularity at $k = 0$. One can prove that this does not spoil the good properties of the spectral transform. In the following section where the matrix spectral problem is reduced to a scalar problem this will be shown explicitly.

6 Lax triplet

6.1 Semi-discrete case

We can write the spectral problem (3.1) in component-form and decouple the two equations to get

$$\chi_{n+2}^{(1)} = g_n \chi_{n+1}^{(1)} - (1+k^2)\chi_n^{(1)} \tag{6.1}$$

$$\chi_n^{(2)} = \frac{1}{ik}\left[\chi_{n+1}^{(1)} - e^{-\frac{i}{2}(\theta_{n+1}-\theta_n)}\chi_n^{(1)}\right] \tag{6.2}$$

with

$$g_n = e^{\frac{i}{2}(\theta_{n+1}-\theta_n)} + e^{-\frac{i}{2}(\theta_{n+2}-\theta_{n+1})} \tag{6.3}$$

i.e. we recover the discrete Schrödinger equation whose scattering theory was given in [9]. In this case the time evolution is fixed by the pair of spectral problems

$$k^2\partial_t\chi_n^{(1)} = -\gamma e^{-i\theta_n}\chi_{n+1}^{(1)} + \gamma e^{-\frac{i}{2}(\theta_{n+1}+\theta_n)}\chi_n^{(1)} \tag{6.4}$$

$$\partial_t\chi_{n+1}^{(1)} = e^{-\frac{i}{2}(\theta_{n+1}-\theta_n)}\partial_t\chi_n^{(1)} + \gamma e^{-i\theta_{n+1}}\chi_n^{(1)}. \tag{6.5}$$

The compatibility requirement for the three spectral problems (6.1), (6.4) and (6.5) furnishes after one integration the semi-discrete sine-Gordon equation (2.8).

Note that, if we introduce

$$\alpha_n = e^{-\frac{i}{2}(\theta_{n+1}-\theta_n)}, \tag{6.6}$$

the equation defining g_n can be rewritten as

$$g_n = \alpha_{n+1} + \frac{1}{\alpha_n}, \tag{6.7}$$

that can be considered as a discrete Riccati equation, which can be solved for α_n in terms of a continuous fraction of g_n (see [10]).

Note also that for θ_n vanishing at large n $(\mathrm{mod}\,\pi)$ we have from (6.3) that $g_n - 2 \to 0$ and the spectral theory developed in [9] is applicable. However, the condition required to be satisfied by $g_n - 2$ in [9] is stricter than (3.44).

We conclude that the Orfanidis matrix spectral problem can be reduced to a scalar Schrödinger spectral problem if the potential $\theta_n \to 0$ $(\mathrm{mod}\,\pi)$ sufficiently rapidly.

6.2 Time evolution of spectral data

The evolution of the spectral data can be determined by considering the first and the third Lax operators, i.e. the spectral problem (6.1) and (6.4).

In order to meet the notations used in [9] we introduce the eigenfunction as follows

$$\chi_n^{(1)}(k) = (1+ik)^n\xi_n(k). \tag{6.8}$$

The principal spectral problem (6.1) reads

$$(1+ik)\xi_{n+2} + (1-ik)\xi_n = g_n\xi_{n+1} \tag{6.9}$$

while the auxiliary spectral problem (6.4) becomes

$$k^2 \partial_t \xi_n = \gamma e^{-\frac{i}{2}(\theta_{n+1}+\theta_n)} \xi_n - (1+ik)\gamma e^{-i\theta_n} \xi_{n+1}. \tag{6.10}$$

We need the Jost solution $\mu_n^+(k)$ introduced in [9], which is analytic in the upper half-plane and is defined via the following discrete integral equation

$$\mu_n^+(k) = 1 - \frac{1}{2ik} \sum_{j=n+1}^{+\infty} \left[1 - \left(\frac{1+ik}{1-ik} \right)^{j-n} \right] (g_{j-1} - 2) \mu_j^+(k) \tag{6.11}$$

and which has the following asymptotic behaviour for $n \to +\infty$

$$\lim_{n \to +\infty} \mu_n^+(k) = 1, \quad k_{\mathrm{Im}} \geq 0 \tag{6.12}$$

and for $n \to -\infty$

$$\mu_n^+(k) \sim a^+(k) + \left(\frac{1-ik}{1+ik} \right)^n b^+(k), \qquad k_{\mathrm{Im}} = 0 \tag{6.13}$$

where $a^+(k)$ is the inverse of the transmission coefficient and $b^+(k)$ is the reflection coefficient.

Then we look for a solution of (6.10) of the form

$$\xi_n(t;k) = \Omega(t;k) \mu_n^+(t;k) \tag{6.14}$$

If we substitute (6.14) into (6.10), we get

$$k^2 \left[\Omega_t \Omega^{-1} \mu_n^+ + \partial_t \mu_n^+ \right] = \gamma e^{-\frac{i}{2}(\theta_{n+1}+\theta_n)} \mu_n^+ - (1+ik)\gamma e^{-i\theta_n} \mu_{n+1}^+. \tag{6.15}$$

If $\theta_n \to r\pi$ ($r \in \mathbb{N}$) for $n \to \pm\infty$, taking into account the asymptotic behaviour of μ_n^+ as $n \to +\infty$ we get, first,

$$\Omega_t \Omega^{-1} = -\frac{i\eta\gamma}{k + i0\eta\gamma t}, \quad \eta = (-1)^r \tag{6.16}$$

and, then, by considering the limit for $n \to -\infty$ ($k_{\mathrm{Im}} = 0$) the evolution equation for the spectral data

$$a_t^+(k) = 0 \tag{6.17}$$

$$b_t^+(k) = \frac{2i\eta\gamma}{k - i0\eta\gamma t} b^+(k), \tag{6.18}$$

which can be trivially integrated to

$$a(t;k) = a(0;k) \tag{6.19}$$

$$b(t;k) = b(0;k) \exp\left[\frac{2i\eta\gamma}{k} t \right]. \tag{6.20}$$

Let us now study in details the behaviour of the Jost solutions at $k = 0$ when the time is switched on. For the sake of simplicity, let us take $\eta\gamma = 1$, i.e. let us scale the time.

In [9] the Riemann-Hilbert boundary value problem defining the Jost solutions was given as

$$\mu_n^-(t;k) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{1-is}{1+is}\right)^n \frac{\mu_n^-(t;-s)\rho^+(t,s)}{s-k+i0}ds \tag{6.21}$$

$$\frac{\mu_n^+(t;k)}{a^+(k)} = 1 + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{1-is}{1+is}\right)^n \frac{\mu_n^-(t;-s)\rho^+(t,s)}{s-k-i0}ds. \tag{6.22}$$

For the sake of simplicity, we omit a possible contribution from the discrete part of the spectrum. It can be added without difficulty along the same lines followed for the Orfanidis spectral problem.

For convenience we introduce

$$\begin{aligned}\Phi_n(k) &= \Omega^{-1}(k)\partial_t\xi_n^-(k) = \Omega^{-1}(k)\left[\Omega(k)\mu_n^-(k)\right]_t \\ &= -\frac{i}{k+i0t}\mu_n^-(k) + \partial_t\mu_n^-(k).\end{aligned} \tag{6.23}$$

Deriving (6.21) with respect to time we have

$$\begin{aligned}\partial_t\mu_n^-(k) = &\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{1-is}{1+is}\right)^n \frac{\mu_n^-(-s)\rho^+(s)}{s-k+i0}\frac{2i}{s-i0t}ds \\ &+ \int_{-\infty}^{+\infty}\left(\frac{1-is}{1+is}\right)^n \frac{\left(\partial_t\mu_n^-(-s)\right)\rho^+(s)}{s-k+i0}ds\end{aligned} \tag{6.24}$$

and therefore

$$\begin{aligned}\Phi_n(k) = &-\frac{i}{k+i0t} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{1-is}{1+is}\right)^n \frac{\mu_n^-(-s)\rho^+(s)}{s-k+i0}\left(\frac{i}{s-i0t} - \frac{i}{k+i0t}\right)ds \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{1-is}{1+is}\right)^n \frac{\Phi_n(-s)\rho^+(s)}{s-k+i0}.\end{aligned} \tag{6.25}$$

Since

$$\frac{1}{s-i0t} - \frac{1}{k+i0t} = \frac{k-s}{(k+i0t)(s-i0t)} \tag{6.26}$$

we have finally

$$\Phi_n(k) = -\frac{1}{k+i0t}S_n(t) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{1-is}{1+is}\right)^n \frac{\Phi_n(-s)\rho^+(s)}{s-k+i0} \tag{6.27}$$

where

$$S_n(t) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{1-is}{1+is}\right)^n \frac{\mu_n^-(-s)\rho^+(s)}{s-i0t} = \begin{cases} \frac{\mu_n^+(0)}{a^+(0)} & \text{for } t>0 \\ \mu_n^-(0) & \text{for } t<0 \end{cases}. \tag{6.28}$$

In order to study the singularity of $\Phi_n(k)$ at $k=0$ one can try to solve the integral equation by iteration and see at each step how the singularity $1/(k-i0t)$ is transformed.

Let us first consider the case $t>0$. Iterating once yields an integral equation that contains the distribution

$$\frac{1}{s-i0}\frac{1}{s-k+i0} = \frac{1}{k-i0}\left(\frac{1}{s-k+i0} - \frac{1}{s-i0}\right). \tag{6.29}$$

Therefore in this case the iteration renormalizes the coefficient of the singularity $1/(k-i0t)$ but does not change the nature of the singularity at $k=0$.

Let us now consider the case $t<0$. In this case in iterating we get in the integral

$$\frac{1}{s+i0}\frac{1}{s-k+i0} \tag{6.30}$$

which is a distribution continuous at $k=0$. Therefore in this case we can conclude that the singularity of $\Phi_n(k)$ can be singled out and it is just

$$-\frac{i}{k-i0t}\mu_n^-(0). \tag{6.31}$$

Recalling the definition of Φ_n we conclude that, for $t<0$, $\partial_t\mu_n^-(k)$, thanks to the continuity of $\mu_n^-(k)$ at $k=0$, is less singular than $1/k$.

In conclusion, $k\partial_t\mu_n^-(k)$ and, therefore, $k^2\partial_t\xi_n^-(k)$ appearing in the auxiliary spectral problem, are continuous at $k=0$. Analogously of course for $k\partial_t\mu_n^+(k)$ and $k^2\partial_t\xi_n^+(k)$. They are, however, discontinuous at $t=0$, i.e. $g_n(t)$ and, consequently, $\theta_n(t)$ evolve according to different laws for $t \lesseqgtr 0$, which is not surprising since the sine-Gordon equation is not an evolution equation.

6.3 Doubly discrete case

Analogously in the system of the two spectral problems (2.1) and (2.2) the two components $\chi_{n,m}^{(1)}$ and $\chi_{n,m}^{(2)}$ of $\chi_{n,m}$ can be decoupled. Of course $\chi_{n,m}^{(1)}$ satisfy the same discrete version of the Schrödinger spectral problem as $\chi_n^{(1)}$, i.e.

$$\chi_{n+2,m}^{(1)} = g_{n,m}\chi_{n+1,m}^{(1)} - (1+k^2)\chi_{n,m}^{(1)} \tag{6.32}$$

where

$$g_{n,m} = e^{\frac{i}{2}(\theta_{n+1,m}-\theta_{n,m})} + e^{-\frac{i}{2}(\theta_{n+2,m}-\theta_{n+1,m})}, \tag{6.33}$$

while the time evolution is fixed by the couple of spectral problems

$$k^2\chi_{n,m+1}^{(1)} = \left(k^2+\gamma e^{-\frac{i}{2}(\theta_{n+1,m}+\theta_{n,m+1})}\right)\chi_{n,m}^{(1)} - \gamma e^{-\frac{i}{2}(\theta_{n,m+1}+\theta_{n,m})}\chi_{n+1,m}^{(1)} \tag{6.34}$$

$$\begin{aligned}\chi_{n+1,m+1}^{(1)} &= \chi_{n+1,m}^{(1)} + e^{-\frac{i}{2}(\theta_{n+1,m+1}-\theta_{n,m+1})}\chi_{n,m+1}^{(1)} \\ &\quad + \left(\gamma e^{\frac{i}{2}(\theta_{n,m+1}+\theta_{n,m})} - e^{-\frac{i}{2}(\theta_{n+1,m}-\theta_{n,m})}\right)\chi_{n,m}^{(1)}.\end{aligned} \tag{6.35}$$

The compatibility among the three spectral problems (6.32), (6.34) and (6.35) furnishes, after an integration, the Hirota-Bianchi doubly discrete sine-Gordon equation.

6.4 Time evolution of spectral data

Following a procedure analogous to that followed in the semi-discrete case we look for a solution of (6.34) of the form

$$\xi_{n,m}(k) = \Omega_m(k)\mu_{n,m}^+(k)(1+ik)^n \tag{6.36}$$

where $\mu^+_{n,m}(k)$ is the Jost solution introduced in [9]. Taking into account the asymptotic behaviours (6.12) and (6.13), if $\theta_{n,m} \to r\pi$ ($r \in \mathbb{N}$) for $n \to \pm\infty$, we obtain, respectively, for $n \to +\infty$ the time evolution of Ω

$$\Omega_{m+1}(k)\Omega_m^{-1}(k) = 1 - \frac{i\eta\gamma}{k} \tag{6.37}$$

and for $n \to -\infty$ the time evolution of the spectral data ($k_{\text{Im}} = 0$)

$$a_{m+1}(k) = a_m(k) \tag{6.38}$$

$$b_{m+1}(k) = -\frac{(\eta\gamma - ik)}{(\eta\gamma + ik)} b_m(k) \tag{6.39}$$

where η is defined as above.

If we introduce

$$a_m = a(t + \tau m), \qquad b_m = b(t + \tau m), \qquad t = n\tau, \tag{6.40}$$

change the constant γ as follows

$$\gamma \to \tau\gamma \tag{6.41}$$

and take the limit $\tau \to 0$ and $n \to \infty$ at t fixed we recover, as expected, the time evolution of the spectral data for the semi-discrete sine-Gordon equation.

6.5 Acknowledgments

This work was partially supported by PRIN 2000 "Sintesi" and was performed in the framework of the INTAS project 99-1782. The authors acknowledge useful critical remarks by an anonymous referee.

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